Math 3236 Statistical Theory

1/17/2023

Law of Large Numbers.

X_i are i.i.d r.v

random Sample

$$\bar{X}_N = \frac{1}{N} \sum_{i=1}^{N} X_i$$

$$\bar{X}_N \xrightarrow[N \to \infty]{P} \mathbb{E}(X_i)$$

$$\forall \varepsilon \quad \mathbb{P}\left(\left| \bar{X}_N - \mathbb{E}(X_i) \right| > \varepsilon \right) \xrightarrow[N \to \infty]{} 0$$

Chebyshev.

X is a r.v. and $\mu = \mathbb{E}(X)$

$$\mathbb{P}(|X - \mu| > \delta) \leq \frac{V(X)}{\delta^2}$$

if X_i are i.i.d. and $\sigma^2 = V(X_i)$

$$\mathrm{Var}(\bar{X}) = \frac{1}{N}\sigma^2.$$

Suppose that $X_N \xrightarrow{P} Y$

$$\forall \varepsilon \quad \mathbb{P}(|X_N - Y| > \varepsilon) \xrightarrow[N \to \infty]{} 0$$

and f is a continuous function

$$f(X_n) \xrightarrow{P} f(Y)$$

Markov Inequality

if X is positive $(\mathbb{P}(X \geq 0) = 1)$

$$\mathbb{P}(X > \delta) \leq \frac{\mathbb{E}(X)}{\delta}$$

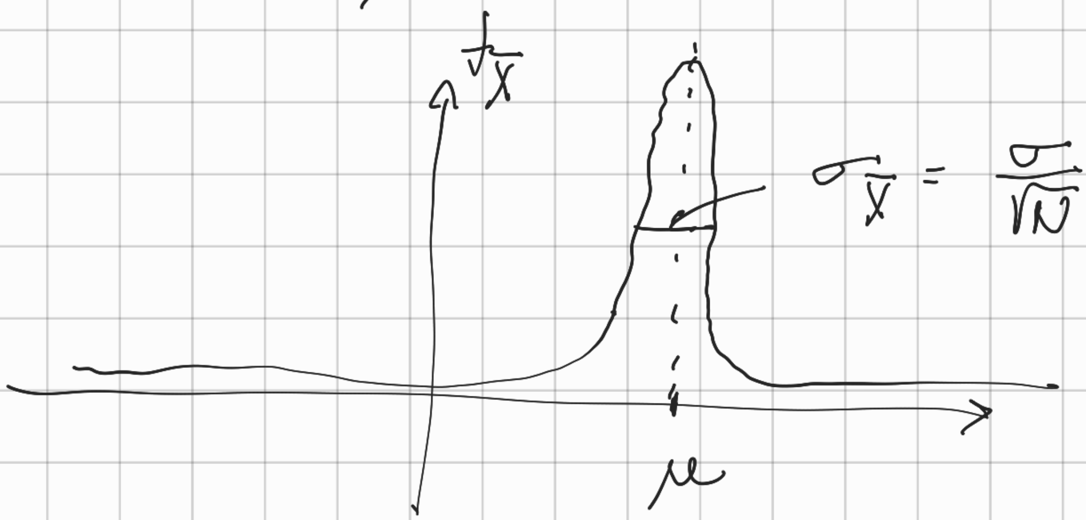
C.L.T. :

X_i is a random sample

$\mathbb{E}(X_i) = \mu$ $\qquad$ $V(X_i) = \sigma^2$

$$\overline{X} = \frac{1}{N} \sum_i X_i$$

$$\mathbb{E}(\overline{X}) = \mu \qquad V(\overline{X}) = \frac{\sigma^2}{N}$$



$$Z_N = \frac{1}{\sqrt{N}} \sum_i \frac{X_i - \mu}{\sigma}$$

$$\mathbb{E}(Z_N) = 0 \qquad V(Z_N) = 1 \qquad \forall N$$

<u>Theorem</u>:

$$Z_N \xrightarrow[N\to\infty]{} Z$$

Z is $\mathcal{N}(0, 1)$

$$\mathbb{P}(Z_N \leq z) \xrightarrow{N\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx$$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx$$

Berry-Essen Inequalities.

$$\left| \mathbb{P}(Z_N \leq z) - \Phi(z) \right| \sim \frac{C(z)}{\sqrt{N}}$$

X_i are random Sample

$\mathbb{E}(X_i) = \mu \qquad V(X_i) = \sigma^2$

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{X_i - \mu}{\sigma} \rightarrow Z \qquad \text{where } Z \text{ is } N(0,1)$$

$$\bar{X} = \frac{1}{N} \sum_i X_i$$

$$\frac{1}{\sqrt{N}} \left(\frac{1}{\bar{X}} - \frac{1}{\mu} \right) \Rightarrow$$

Delta method

Y_n is a sequence of r.v.

a_n is a sequence of real number

$\lim_{n \to \infty} a_n = \infty$

θ is a real number

$$a_n (Y_n - \theta) \xrightarrow{d} F^*$$

(previos case

$$Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad X_i \quad i.i.d$$

$$\theta = \mu$$

$$a_n = \frac{\sqrt{n}}{\sigma}$$

$$a_n (Y_n - \theta) = \frac{\sqrt{n}}{\sigma} \left(\frac{1}{n} \sum_{i=1}^{n} X_i - \mu \right) =$$

$$= \frac{1}{\sqrt{n}} \left(\sum_i \frac{X_i - \mu}{\sigma} \right))$$

$a(x)$ is a differentiable function

$$a_n\left(\alpha(Y_n)-\alpha(\theta)\right)$$

$a_n(Y_n-\theta) \rightarrow$ converge to a dist.

$a_n(Y_n-\theta)$ is almost always finite

$a_n \to \infty \Rightarrow Y_n-\theta$ must be small.

$$\alpha(Y_n) \approx \alpha(\theta)+\alpha'(\theta)(Y_n-\theta)+\dots$$

$$a_n\left(\alpha(Y_n)-\alpha(\theta)\right) \approx a_n\,\alpha'(\theta)(Y_n-\theta)$$

$$\frac{a_n}{\alpha'(\theta)}\left(\alpha(Y_n)-\alpha(\theta)\right) \approx a_n(Y_n-\theta)$$

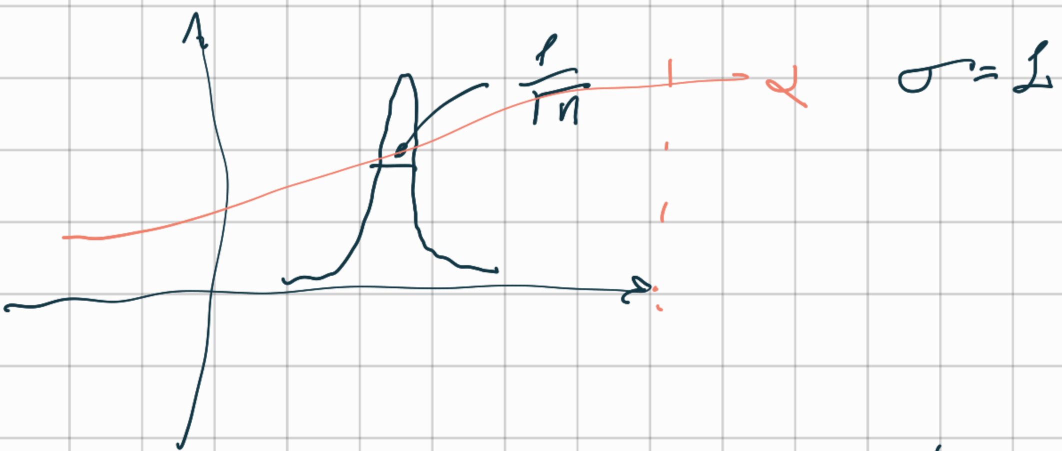
$$\lim_{n\to\infty} \frac{a_n}{\alpha'(\theta)}\Big(\alpha(Y_n) - \alpha(\theta)\Big) = F^*$$

$$\alpha(x) = \frac{1}{x}$$

$$\alpha'(\mu) = -\frac{1}{\mu^2}$$

$$-\frac{n^{1/2}\mu^2}{\sigma}\left(\frac{1}{\overline{X}_n} - \frac{1}{\mu}\right) \to \mathcal{N}(0,1)$$

$$\frac{1}{\overline{X}_n} - \frac{1}{\mu} \simeq \mathcal{N}\left(0, \frac{\sigma^2}{n\mu^4}\right)$$



$$\frac{1}{n}\sum_i X_i = \overline{X} \qquad \sqrt{n}\left(\overline{X} - \mu\right) = Z_n$$

$$Z_n = \frac{1}{\sqrt{n}}\sum_i \left(X_i - \mu\right)$$